MATCHING PRECLUSION AND CONDITIONAL MATCHING PRECLUSION PROBLEMS FOR TWISTED CUBES

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ABSTRACT. The matching preclusion number of a graph is the minimum number of edges whose deletion results in a graph that has neither perfect matchings nor almost-perfect matchings. For many interconnection networks, the optimal sets are precisely those induced by a single vertex. Recently, the conditional matching preclusion number of a graph was introduced to look for obstruction sets beyond those induced by a single vertex. It is defined to be the minimum number of edges whose deletion results in a graph *with no isolated vertices* that has neither perfect matchings nor almost-perfect matchings. In this paper, we find the matching preclusion number and the conditional matching preclusion number for twisted cubes, an improved version of the well-known hypercube. Moreover, we also classify all the optimal matching preclusion sets.

Keywords: Interconnection networks, twisted cubes, perfect matching

1. INTRODUCTION AND PRELIMINARIES

A *perfect matching* in a graph is a set of edges such that every vertex is incident to exactly one edge in this set. An *almost-perfect matching* in a graph is a set of edges such that every vertex, except one, is incident to exactly one edge in this set, and the exceptional vertex is incident to none. So if a graph has a perfect matching, then it has an even number of vertices; if a graph has an almost-perfect matching, then it has an odd number of vertices. The *matching preclusion number* of a graph G, denoted by $mp(G)$, is the minimum number of edges whose deletion leaves the resulting graph without a perfect matching or almost-perfect matching. Any such optimal set is called an *optimal matching preclusion set*. The deleted edges may be referred to as *faults* or *faulty edges* and the resulting graph as a *faulty graph*. We define $mp(G) = 0$ if G has neither a perfect matching nor an almost-perfect matching. This concept of matching preclusion was introduced

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by [1] and further studied by [2, 4, 6, 7]. They introduced this concept as a measure of robustness in the event of edge failure in interconnection networks, as well as a theoretical connection to conditional connectivity, "changing and unchanging of invariants" and extremal graph theory. We refer the readers to [1] for details and additional references. We use standard graph theory terminology in this paper.

Useful distributed processor architectures offer the advantage of improved connectivity and reliability. An important component of such a distributed system is the system topology, which defines the inter-processor communication architecture. In certain applications, every vertex requires a special partner at any given time and the matching preclusion number measures the robustness of this requirement in the event of edge failures as indicated in [1]. Hence in these interconnection networks, it is desirable to have the property that the only optimal matching preclusion sets are those whose elements are incident to a single vertex.

Proposition 1.1. *Let G be a graph with an even number of vertices. Then $mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

Proof. Deleting all edges incident to a single vertex will give a graph with no perfect matchings and the result follows. □

We call an optimal solution of the form given in the proof of Proposition 1.1 a *trivial optimal matching preclusion set.* As mentioned earlier, it is desirable for an interconnection network to have only trivial optimal matching preclusion sets. A graph G is *super matched* if $mp(G) = \delta(G)$ and every optimal matching preclusion set is trivial. Given that it is unlikely that in the event of random edge failure, all of these failures will be at the same vertex, it is natural to ask what the next obstruction sets are for a graph with edge failure to have a perfect matching subject to the condition that the faulty graph, the graph created with the faulty edges deleted, has no isolated vertices. This motivates the definition given in [3]. The *conditional matching preclusion number* of a graph G, denoted by $mp_1(G)$, is the minimum number of edges whose deletion leaves the resulting graph with no isolated vertices and without a perfect matching or almost-perfect matching. Any such optimal set is called an *optimal conditional matching preclusion set.* We define $mp_1(G) = 0$ if G has neither a perfect matching nor an almost-perfect matching. We will leave $mp_1(G)$ undefined if a conditional matching preclusion set does not exist; that is, if we cannot delete edges to satisfy both conditions in the definition.

If we delete edges so that the resulting graph has no isolated vertices, then a basic obstruction to a perfect matching will be the existence of a path $u - v - w$ in the resulting graph, where the degree of u and the degree of w are 1. So to produce such an obstruction set, one can pick any path

$u - v - w$ in the original graph and delete all the edges incident to either u or w but not v. We define $\nu_e(G)$ to be

$$\min\{d_G(u) + d_G(w) - 2 - y_G(u, w) : \text{there is a 2-path from } u \text{ to } w\}$$

where $d_G(u)$ is the degree of vertex u and $y_G(u, w) = 1$ if u and w are adjacent, and 0 otherwise. (We will suppress G and simply write d and y if it is clear from the context.) So mirroring Proposition 1.1, we have the following result.

Proposition 1.2. *Let G be a graph with an even number of vertices. Suppose every vertex in G has degree at least three. Then*

$$mp_1(G) \leq \nu_e(G).$$

We call an optimal solution of the form induced by ν_e a *trivial optimal conditional matching preclusion set.* As mentioned earlier, the matching preclusion number measures the robustness of this requirement in the event of link failures, so it is desirable for an interconnection network to be super matched. Similarly, it is desirable to have the property that the only optimal conditional matching preclusion sets are trivial as well. Such an interconnection network is *conditionally super matched.* [3] introduced this concept and considered the conditional matching preclusion problem for a number of basic networks including the hypercubes and it was proved that they have this desired property.

The vertex set of the *twisted n-cube* TQ_n, is the set of all binary strings of length n when $n \geq 1$ is odd. Consider the vertex $u = u_{n-1}u_{n-2}\ldots u_1u_0$ in TQ_n. For $0 \leq i \leq n-1$, we define the *ith parity function* $P_i(u) = u_i \oplus u_{i-1} \oplus \cdots \oplus u_0$, where $\oplus$ is the exclusive-or operation. That is, $P_i(u) = 1$ if $u_iu_{i-1}\cdots u_0$ has an odd number of 1's and $P_i(u) = 0$ if $u_iu_{i-1}\cdots u_0$ has an even number of 1's. A twisted cube is defined recursively for TQ_n as follows: A twisted 1-cube, TQ_1, is a graph with two vertices 0 and 1 that are adjacent to each other. Suppose that $n \geq 3$. The vertex set of TQ_n is decomposed into four sets, $00T_{n-2}$, $01T_{n-2}$, $10T_{n-2}$, $11T_{n-2}$, where ijT_{n-2} consists of those vertices u with $u_{n-1} = i$ and $u_{n-2} = j$. For each $(i,j) \in \{(0,0),(0,1),(1,0),(1,1)\}$, the subgraph of TQ_n induced by ijT_{n-2}, denoted by $ijTQ_{n-2}$, is isomorphic to TQ_{n-2}. These $ijTQ_{n-2}$'s are the *twisted subcubes.* Moreover, there are edges between different twisted subcubes as follows: A node $u = u_{n-1}u_{n-2}\cdots u_1u_0$ with $P_{n-3}(u) = 0$ is adjacent to $v = v_{n-1}v_{n-2}\cdots v_1v_0$, where $v = \bar{u}_{n-1}\bar{u}_{n-2}u_{n-3}\cdots u_1u_0$ or $v = \bar{u}_{n-1}u_{n-2}u_{n-3}\cdots u_1u_0$; a node u with $P_{n-3}(u) = 1$ is adjacent to v, where $v = u_{n-1}\bar{u}_{n-2}u_{n-3}\cdots u_1u_0$ or $v = \bar{u}_{n-1}u_{n-2}u_{n-3}\cdots u_1u_0$. These are called the *cross edges.* See Figure 2 for TQ_1 and TQ_3. For convenience, we may refer to a vertex as *even* if it has an even number of 1's is and *odd* if it has an odd number of 1's. It is clear from the definition that each vertex is incident to two cross edges. Moreover, every vertex has a neighbor of different parity. See Figure 1. We note that there are two types of 4-cycles formed by the cross edges. If we draw one in planar form, then some will have the same form and other will have a "twisted" form, depending on the parity of $P_{n-3}(u)$. Indeed, it follows from the definition that every vertex in TQ_n has at least $(n+1)/2$ vertices of opposite parity. It is obvious that TQ_n has 2^n vertices and is n-regular. Moreover, there are at least 2^{n-3} independent edges between two twisted subcubes in TQ_n.

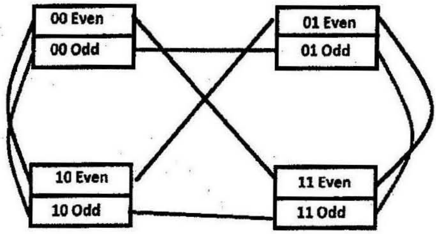

FIGURE 1. Cross edges forming a cycle

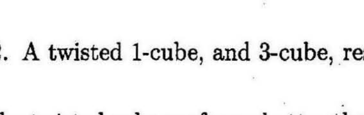

FIGURE 2. A twisted 1-cube, and 3-cube, respectively

As shown in [5], the twisted cube performs better than that of the hypercube in terms of robustness and strength. In fact, all properties that can be considered for both cubes, listed in [1, 4, 7], show a stronger and more reliable network for the twisted cube. Thus, the twisted cube proves to be an attractive alternative to that of the more popular hypercube. In this paper, we need the following result on Hamiltonicity of the twisted cubes. (A graph is *Hamiltonian connected* if there is a Hamiltonian path between every pair of vertices.)

Theorem 1.3. *[5] Let $n \geq 3$. If $F \subseteq V(TQ_n) \cup E(TQ_n)$ where $|F| \leq n-2$, then $TQ_n - F$ is Hamiltonian. If $F \subseteq V(TQ_n) \cup E(TQ_n)$ where $|F| \leq n-3$, then $TQ_n - F$ is Hamiltonian connected.*

2. Matching Preclusion

Theorem 2.1. *Let $n \geq 1$ be odd. Then $mp(TQ_n) = n$*

Proof. It follows from Proposition 1 that $mp(TQ_n) \leq n$, so now we want to prove equality. We proceed by induction. It is trivial to check that $mp(TQ_1) = 1$ and $mp(TQ_3) = 3$. Now suppose that $mp(TQ_k) = k$ for some $k \geq 1$. Consider the graph TQ_{k+2} where $k \geq 3$. We decompose TQ_{k+2} into $00TQ_k, 01TQ_k, 10TQ_k$ and $11TQ_k$ via the recursive structure. Now, by way of contradiction, assume that $mp(TQ_{k+2}) < k+2$. Then we can delete some set of $k+1$ edges, F, so that $TQ_{k+1} - F$ has no perfect matchings. Note that for every k-bit string x, the elements $00x, 01x, 10x, 11x$ lie on a 4-cycle:

$$(00x, 01x, 11x, 10x) \text{ if } x \text{ is odd and}$$
$$(00x, 11x, 01x, 10x) \text{ if } x \text{ is even.}$$

Call a 4-cycle of this form *reaching*. Hence among all 2^k k-digit strings x, we have a reaching 4-cycle containing the elements in $\{00x, 01x, 10x, 11x\}$. Also, by the recursive decomposition of TQ_{k+2}, every cross edge is in exactly one reaching 4-cycle, so the cross edges of TQ_{k+2} can be partitioned into 2^k 4-cycles. Thus, to destroy all perfect matchings in TQ_{k+2}, we need to delete at least two neigboring cross edges from the same reaching 4-cycle, since otherwise, we would have a perfect matching by picking two appropriate edges from each of the 2^k reaching 4-cycles. Hence, at most $k-1$ elements of F are not cross-edges. Let $F_{ij} = F \cap E(ijTQ_k)$ for every (i,j). Then by the induction hypothesis, each $ijTQ_k - F_{ij}$ has a perfect matching. Therefore, $TQ_{k+2} - F$ has a perfect matching, a contradiction. □

Theorem 2.2. *Let $n \geq 5$ be odd. Then TQ_n is super matched.*

Proof. Again, we proceed by induction. A computer check is applied for TQ_5. (We note that the statement is not true for $n = 3$; see Figure 4 where the dashed lines correspond to faulty edges.) Now consider TQ_{k+2} where $k \geq 5$. Let F be an optimal matching preclusion set. So $|F| = k+2$ by Theorem 2.1. Now $TQ_{k+2} - F$ has no perfect matching. As in the previous proof, F must contain two neigboring cross edges from some reaching 4-cycle. For notational convenience, we assume that these two edges are incident to the same vertex in $00TQ_k$, say $00x$, where x is a k-digit string. (Our proof will not explicitly use the fact that this is in $00TQ_k$; it is general enough that such an assumption is only for notational convenience.) Define F_{ij} as before. Since F is a matching preclusion set, at least one of $ijTQ_{ij} - F_{ij}$ has no perfect matchings. Since $mp(TQ_k) = k$, we may conclude that F contains exactly two cross edges and the remaining k edges from F must belong to exactly one twisted subcube. We now consider 2 cases.

Case 1: The remaining k faulty edges are deleted from a set other than $00TQ_k$. So they are in $ijTQ_k$ where $(i, j) \neq (0, 0)$. By the induction hypothesis, these k edges in F_{ij} must all be incident to the same vertex, say ijy. Now F contains exactly two faulty cross edges and they are incident to $00x$ as assumed earlier. So every edge in F is incident to either $00x$ or $01y$. Pick any neighbor of $00x$ in $00TQ_k$, say $00u$. Then $(00x, 00u) \notin F$. Now ijy is incident to two cross edges and at most one of them is incident to $00x$. Let (ijy, z) be a cross edge such that $z \neq 00x$. Thus it is not in F. Suppose $k \geq 5$ and so $k+2 \geq 7$. Now $TQ_{k+2} - \{00x, 00u, ijy, z\}$ does not contain elements of F. It is easy to see that $00x, 00u, ijy, z$ are distinct. By Theorem 1.3 as $k + 2 - 2 \geq 5 > 4$, $TQ_{k+2} - \{00x, 00u, ijy, z\}$ has an even Hamiltonian cycle, and hence a perfect matching M. Now, $M \cup \{(00x, 00u), (ijy, z)\}$ is a perfect matching in $TQ_{k+1} - F$, a contradiction.

Case 2: The remaining k faulty edges are deleted from $00TQ_k$. Again, by the induction hypothesis, they are incident to the same vertex, say $00y$. If $00x = 00y$, then we are done. So assume $x \neq y$. Pick a neighbor of $00y$, say $00z$ where $x \neq z$. We note that $(00y, 00z) \in F$. Clearly, the cross edges incident to $00y$ or $00z$ are not in F. It is easy to see that we can choose two of these cross edges, one from each vertex, such that the other end vertices are in the same twisted subcube $ijTQ_k$ where $(i, j) \neq (0, 0)$. Let the vertices in $ijTQ_k$ incident to these cross edges be ijy and ijz, respectively. If $(i', j') \notin \{(0, 0), (i, j)\}$, then $F_{i'j'} = \emptyset$, and so $i'j'TQ_{k+2} - F_{i'j'}$ has a perfect matching say $M_{i'j'}$. By Theorem 1.3, there is a Hamiltonian path P_{00} in $00TQ_k$ between $00y$ and $00z$ and a Hamiltonian path P_{ij} in $ijTQ_k$ between ijy and ijz. Clearly P_{00} contains exactly one element in F and it must be incident to $00y$. Since $F_{ij} = \emptyset$, P_{ij} contains no elements of F. So they induce perfect matchings M_{00} and $M_{i'j'}$ in $00TQ_k - \{00y, 00z\}$ and $ijTQ_k - \{ijy, ijz\}$, respectively. Now $M_{01} \cup M_{00} \cup M_{10} \cup M_{11} \cup \{(00y, ijy), (00z, ijz)\}$ is a perfect matching in $TQ_{k+2} - F$, a contradiction. □

3. Conditional Matching Preclusion Number

Theorem 3.1. *Let $n \geq 5$. Then $mp_1(TQ_n) = 2n - 2$.*

Proof. Since $mp_1(TQ_n) \leq 2n - 2$ by Proposition 1.2, it is enough to prove $2n - 2$ is best possible. We proceed by induction. The base case $n = 5$ was checked using a computer search. So assume $n \geq 7$. We need to show that deleting $2n - 3$ edges in TQ_n either gives us a perfect matching or an isolated vertex. Again we use the recursive structure. As seen in previous proofs, the $ijTQ_{n-2}$'s play a similar role, so for notational simplicity, we will refer to them as T_1, T_2, T_3, T_4. (Obviously, we can only use properties that apply to all of the $ijTQ_{n-2}$'s.) Let $F \subseteq E(TQ_n)$ such that $|F| \le 2n-3$, $F_i = F \cap (E(T_i))$ with $t_i = |F_i|$ and F_c be the set of cross edges in F. As before, $|F_c| \ge 2$ and F_c contains a pair of adjacent edges in a reaching 4-cycle. We may assume at least one of $T_1 - F_1, T_2 - F_2, T_3 - F_3, T_4 - F_4$ has no perfect matchings, or else we are done. Therefore, we may assume that it is $T_1 - F_1$ for notational simplicity, which implies that $t_1 \ge n-2$. Hence, $t_2, t_3, t_4 \le n-3$, so each of $T_2 - F_2, T_3 - F_3, T_4 - F_4$ has a perfect matching. Since $|F_c| \ge 2$, then $t_1 \le 2n-5$. We consider 3 cases.

Case 1: $n-2 \le t_1 \le 2n-7$. We note that $t_2, t_3, t_4 \le 2n-3-(n-2)-2 = n-3$. By the induction hypothesis, there exists an isolated vertex in T_1, say v. It is clear that v is the unique isolated vertex in $T_1 - F_1$. We may assume that at least one of the cross edges incident to v is not faulty. Otherwise, v is an isolated vertex in $TQ_n - F$, and we are done. Thus, suppose that this non-faulty cross edge is (v, w), where w is in T_i such that $i = 2, 3$, or 4. Note that there are at least 2^{n-3} edges between T_1 and T_i. Since $t_1 \ge n-2$, $|F_c| \le (2n-3) - (n-2) = n-1$. Moreover, $n-1 < 2^{n-3} - 1$ for $n \ge 7$. Therefore, it is impossible for all of the cross edges between T_1 and T_i to be faulty. Pick one of the non-faulty cross edges different from (v, w), and label it (a, b) such that a is in T_1 and b is in T_i. Let F_1' be the set of edges obtained from F_1 by deleting edges in T_1 that are incident to v. We now consider $T_1 - F_1'$. Thus, $|F_1'| \le 2n-7-(n-2) = n-5 = (n-2)-3$. Therefore by Theorem 1.3, $T_1 - F_1'$ is Hamiltonian connected. So there is a Hamiltonian path between v and a in $T_1 - F_1'$. This path contains exactly one element in F, namely, the edge incident to v. So it induces a perfect matching M_1 of $T_1 - \{v, a\}$ that does not contain elements of F. Similarly, we can get a perfect matching M_i of $T_i - \{w, b\}$ that does not contain elements of F as $|F_i| \le n-3$. For $j \notin \{1, i\}$, we can get a perfect matching M_j of T_j. Now, $M_1 \cup M_2 \cup M_3 \cup M_4 \cup \{(v, w), (a, b)\}$ is a perfect matching for $TQ_n - F$.

Case 2: $t_1 = 2n-6$. We note that there can be at most one isolated vertex in $T_1 - F_1$. Thus, we can break this case into 2 subcases.

Subcase 1: There exists one isolated vertex in T_1. Let v be the isolated vertex. Let $F_1' = F_1 - (u, v)$, where u is a neighbor of v in T_1 such that u has no faulty cross edges. (Such an edge exists as $|F_c| \le 3$ and v has $n-2 \ge 5$ neighbors in T_1.) Consider $T_1 - F_1'$. Then, by the induction hypothesis, there exists a perfect matching M_1 in $T_1 - F_1'$ as $T_1 - F_1'$ has no isolated vertices and $|F_1'| = 2n-6-1 = 2n-7 \le 2(n-2)-3$. Moreover, M_1 contains (u, v). We may assume v is incident to a cross edge not in F, or else v is an isolated vertex in $TQ_n - F$ and we are done. Assume, without loss of generality, that the cross edge is incident to a vertex in T_2, say v'. Since u incident to no faulty cross edges and there are two such cross edges, we may assume, for notational simplicity, that it is between u and u' and

u' is a vertex in T_3. We note that $|F_c| + t_2 + t_3 + t_4 = 3$ and $|F_c| \geq 2$. Clearly, we can find a cross edge (x, y) between T_2 and T_3 such that it is not in F, x is in T_2, $x \neq v'$ and $y \neq u'$. Since $|F_2| \leq 1 \leq (n-2) - 3$, by Theorem 1.3, there is a Hamiltonian path between x and v' in T_2, and hence there is a perfect matching M_2 of $(T_2 - F_2) - \{x, v'\}$. Similarly, there is a perfect matching M_2 of $(T_3 - F_3) - \{y, u'\}$. Clearly, there is a perfect matching M_4 of $T_4 - F_4$. Now $M_1 \cup M_2 \cup M_3 \cup M_4 \cup \{(v, v'), (u, u'), (x, y)\}$ is a perfect matching of $TQ_n - F$.

Subcase 2: There exist no isolated vertices in T_1. Consider a faulty edge (u, v) in T_1 such that u is not incident to any cross edge in F. (Such edge must exist as $|F_c| \leq 3$ and $t_1 = 2n - 6 \geq 9$.) Let $F_1' = F_1 - (u, v)$ and consider $T_1 - F_1'$. By the induction hypothesis, there exists a perfect matching in $T_1 - F_1'$. Furthermore, the perfect matching must include (u, v). So we have a perfect matching of $(T_1 - F_1) - \{u, v\}$. We can now proceed as in Subcase 1.

Case 3: $t_1 = 2n - 5$. Here, it is possible for $T_1 - F_1$ to have two isolated vertices. Since $t_1 = 2n - 5$, $|F_c| = 2$ and $t_2 = t_3 = t_4 = 0$. Moreover, both cross edges in F are incident to the same vertex, say v. Since the set of cross edges form a collection of reaching 4-cycles that spans the vertices of TQ_n, we will use them to start the construction of a perfect matching of $TQ_n - F$. Consider the reaching 4-cycle containing v. Let x, y, z be the three vertices that form such a 4-cycle C_1 of cross edges with v. We may assume the cycle is (v, x, z, y). For this cycle, we pick the two edges (v, x) and (y, z). Of course, we have to make an adjustment, as $(v, x) \in F$. We consider two cases, depending on the location of v.

Subcase 1: v is in T_1. Clearly, v must have some neighbor u in $T_1 - F_1$ such that $(u, v) \notin F$, or else we have an isolated vertex, and we are done. Since T_1 is isomorphic to T_2, T_3, and T_4, we have a neighbor u' in the same set as x that corresponds to our u in T_1. Let a and a' be the other vertices that form a reaching 4-cycle C_2 of cross edges with u and u'. If u is adjacent to u', then a is adjacent to a'. So we can use $(u, v), (u', x), (a, a'), (y, z)$ to match up vertices in C_1 and C_2. Now we are free to complete our perfect matching by choosing edges from the other reaching 4-cycles. Thus, assume that u is not adjacent to u'. Since all the faulty edges are either in F_1 or incident to v, we can find two vertices, c and c', that are adjacent to a and a', respectively, such that c is adjacent to c' through their cross edge. (Here, we want the reaching 4-cycle containing a and the one containing c are of different type. Since $n \geq 7$, a has at least 3 neighbors of parity different from a, one of which is y. So such a c exists.) We consider the reaching 4-cycle C_3 of cross edges containing c and c'. Without loss of generality, let the cycle go through vertices d and d' with d in T_1 and d' in T_2. We can now use $(u, v), (u', x), (a, c), (a'c), (y, z), (d, d')$ to matched up vertices in C_1, C_2 and C_3. See Figure 3. We can complete our perfect matching by choosing edges from the other reaching 4-cycles. Hence we are done.

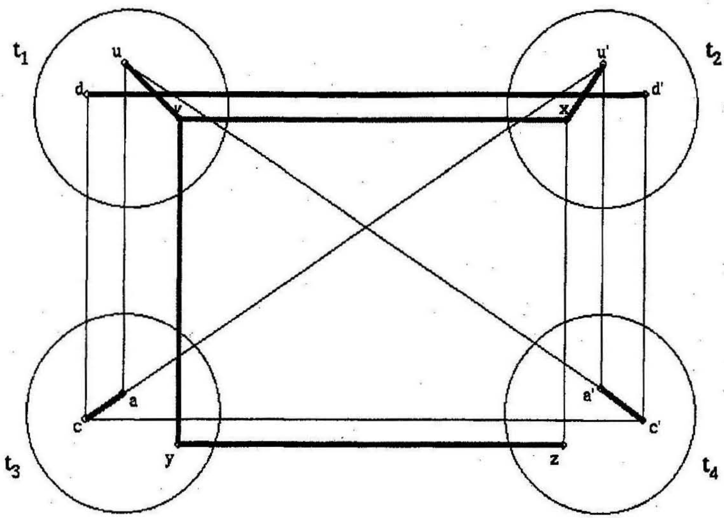

FIGURE 3. A perfect matching where v is in T_1

Subcase 2: v is not in T_1. Then we may assume either x in T_1 or z in T_1. Let w be this vertex in T_1 (either x or y). In either case, if w is not isolated in T_1, then pick w' such that (w, w') is not in F. Let v', z', x', y' be the corresponding vertices. Note that they are vertices of a reaching 4-cycle. Now use $(v, v'), (x, x'), (y, y'), (z, z')$ and it is now easy to extend this to a perfect matching of $TQ_n - F$ using cross edges. So we may assume w is isolated in T_1. Pick any neighbor of w, say w'. Let C_1 be the reaching 4-cycle containing v and C_2 be reaching 4-cycles containing w'. Suppose they are of the same type. Then C_1 and C_2 together with $(v, v'), (x, x'), (y, y'), (z, z')$ form Q_3, the hypercube of dimension 3. Then the faulty edges in this graph are $(v, x), (v, y), (w, w')$ and it has no isolated vertices if these edges are deleted. So it is easy to see that we can match up the vertices $v, v', x, x,', z, z', w, w'$ using non-faulty edges from this graph. We can now extend this to a perfect matching of $TQ_n - F$ using cross edges. So we may suppose C_1 and C_2 are of different parity. Call the graph consisting of C_1 and C_2 and $(v, v'), (x, x'), (y, y')(z, z')$ H. It is easy to check that the only configuration in which there are no perfect matchings in the graph obtained from deleting H is the one in Figure 4. Now suppose w'

has a neighbor h', with opposite parity of w', in T_1 such that $(w', h') \notin F$. Let C_3 be the reaching 4-cycle containing h'. It is not difficult to check that we can match up the vertices using the non-fault edges of the graph induced by the vertices in C_1, C_2, C_3. We know that there are at least $(n-2+1)/2 - 1 = (n-3)/2$ of possible $h' \neq w$. So we may assume that they are in F. Note that (w, v) is an edge. Now $T_1 - \{w, w'\}$ has at most $2n-5-(n-3)/2-(n-2) = (n-3)/2$. Since (w, v) is an edge, $TQ_n - \{w, w'\}$ has at most $(n-3)/2+1$ element of F. (Note that we have included the other cross edge incident to v that are in F.) Now, find s and s' in TQ_n such that (w, s) and (w', s') are not in F. (They exist because $TQ_n - F$ has no isolated vertices and TQ_n has no triangles.) Now let F' be the set of edges in F that is incident to none of w, w', s, s' together with the vertices w, w', s, s'. Then $|F'| \leq (n-3)/2+4 = (n+5)/2$. Now for $n \geq 9$, $n-2 \geq (n+5)/2$. So $TQ_n - F'$ has a Hamiltonian cycle. It may contain one element of F, namely, the cross edge that is incident to v but not w. Hence it induces a perfect matching in $(TQ_n - \{w, w', s, s'\}) - F$, together with (w, s) and (w', s') give a perfect matching of $TQ_n - F$. This leaves the case $n = 7$ to be checked separately. Indeed, one can the proof given here to one that cover the cases $n \geq 7$ but it is more involved so we will just provide a brief description. We also consider an h' such that $(w', h') \notin F$ where w' and h' have the same parity. Then we consider a neighbor of h', say h'', such that h'' and h' are of different parity and $w \neq h''$. We note that h'' exists as h' has $(n-1)/2 - 1$ such neighbors. It is possible that $(h', h'') \in F$. Let C_3 be the reaching 4-cycle containing h' and C_4 be the reaching 4-cycle containing h''. Then one can check that we can match up the vertices using the non-fault edges of the graph induced by the vertices in C_1, C_2, C_3, C_4, and we can proceed as before. So every element of F_1 is incident to w or w'. (Recall that $(w, w') \in F$.) Now $TQ_n - \{w, w'\}$ contains only one element of F and we can proceed as before.

□

4. Conclusion

The class of twisted cubes was introduced as a competitive model to the class of hypercubes. It has been shown that the twisted cubes have many of the good properties that make the hypercubes popular. Indeed, research shows they are superior to the hypercubes in many way. However, many proofs for the twisted cubes are considerably more difficult than the hypercubes for the corresponding properties. In this paper, we show that the twisted cubes are super matched and find their conditional matching preclusion number. Based on data on small cases, we conjecture that they are conditionally super matched except for small cases.

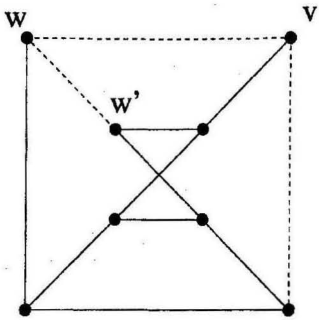

FIGURE 4. An exceptional case

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